

Aspects of Anomalies in Field Theory¹

Kazuo Fujikawa

*Department of Physics, University of Tokyo
Bunkyo-ku, Tokyo 113, Japan*

Abstract

We discuss some formal aspects of quantum anomalies with an emphasis on the regularization of field theory. We briefly review how ambiguities in perturbation theory have been resolved by various regularization schemes. To single out the true quantum anomaly among ambiguities, the combined ideas of PCAC, soft pion limit and renormalizability were essential. As for the formal treatment of quantum anomalies, we mainly discuss the path integral formulation both in continuum and lattice theories. In particular, we discuss in some detail the recent development in the treatment of chiral anomalies in lattice gauge theory.

1 Introduction

The notion of quantum anomalies played key roles in various applications of modern field theory. The good accounts of these developments and applications are found in [1] [2] [3]. Here we discuss some formal aspects of quantum anomalies with an emphasis on the regularization of field theory.

It is well known that the modern field theory was formulated by Tomonaga, Schwinger and Feynman. This modern field theory with renormalization showed that the principles of quantum mechanics are applicable to quite a wide range of phenomena, covering all the energies presumably up to Planck scales. In the early days of modern renormalization theory, however, the treatment of ultraviolet divergences was a problematic procedure.

2 Ambiguities in loop diagrams in field theory

In a letter of Tomonaga to Oppenheimer, which reported on a summary of renormalization theory in Japan and which was later published in Physical Review[4], Tomonaga mentioned a certain difficulty to preserve the gauge invariance of the quadratically divergent vacuum polarization tensor

$$\partial^\mu \langle T^* V_\mu(x) V_\nu(y) \rangle = 0 \quad (2.1)$$

¹Talk given at International Conference on Fundamental Sciences: Mathematics and Theoretical Physics, Singapore, 13-17 March 2000 (To be published in the Proceedings, World Scientific, Singapore)

in interaction picture perturbation theory. Here $V_\mu(x)$ stands for the fermionic vector current to which the photon couples. This one-loop diagram for the self-energy of the electromagnetic field in fact remained as one of the most subtle Feynman diagrams until the developments of modern gauge theory starting around early 1970s.

Motivated by this problematic aspect of the vacuum polarization tensor [5], two members of Tomonaga School, Fukuda and Miyamoto, examined the next simplest diagram, namely the triangle diagram corresponding to the process $\pi^0 \rightarrow \gamma\gamma$. They evaluated the correlation functions (in the modern notation)

$$\langle T^* A_\mu(x) V_\alpha(y) V_\beta(z) \rangle \quad (2.2)$$

and

$$\langle T^* P(x) V_\alpha(y) V_\beta(z) \rangle. \quad (2.3)$$

Here the axial vector current $A_\mu(x)$ and the pseudoscalar density $P(x)$ naively satisfy the relation

$$\partial^\mu A_\mu(x) = 2miP(x) \quad (2.4)$$

where m stands for the fermion mass. But Fukuda and Miyamoto found the violation of gauge invariance in the AVV diagram and also a deviation from the naive relation (2.4) for the two amplitudes by an explicit evaluation in perturbation theory[6].

Apparently, Tomonaga was much interested in this discrepancy and examined the same calculations in (2.2) and (2.3) by using the regularization of Pauli and Villars[7], of which preprint was sent to Tomonaga by Pauli. Tomonaga together with his associates concluded that the gauge invariance of the AVV diagram is maintained by the regularization but the above deviation from the naive relation (2.4) was not uniquely resolved by the Pauli-Villars regularization[8].

Steinberger at Princeton, who learned the calculation by Fukuda and Miyamoto through Yukawa (as is noted in a footnote to his paper[9]), also applied the brand-new Pauli-Villars regularization to the calculations of various decay modes of meson fields. He also arrived at a conclusion[9] similar to that of Tomonaga[8]. The application of the regularization prescription of Pauli and Villars, though it maintained the gauge invariance of the AVV amplitude, appeared to modify the finite part of AVV amplitude and thus the physical life-time of the neutral pion.

In 1951, Schwinger also examined the entire issue of gauge invariance including the vacuum polarization diagram and also the above triangle diagrams[10]. He used the so-called proper time regularization and partly *imposed* the gauge invariance. By this way he successfully handled the vacuum polarization diagram, but the discrepancy in the triangle diagrams remained.

It is interesting that Feynman, unlike Tomonaga and Schwinger, was apparently not much interested in the above subtle behavior of the triangle diagrams. This is presumably due to the fact that Feynman was more interested in the totality of Feynman diagrams rather than in the naive canonical manipulations such as (2.4).

To summarize the analyses of the vacuum polarization diagram and the triangle diagrams around 1950, Nishijima[11] once mentioned that it was difficult to distinguish a subtle difference between the “ambiguity” in perturbation theory and the true “anomaly” at that time.

3 Anomalies and regularization in field theory

In 1969, Bell and Jackiw[12] at CERN and Adler[13] at Princeton analyzed the triangle diagrams in greater detail. The new ingredients in the analysis of Bell and Jackiw were the partially conserved axial-vector current (PCAC) and the picture of the pion as the Nambu-Goldstone particle[14]. Bell and Jackiw noticed the inevitable deviation from PCAC if one applies the conventional Pauli-Villars regularization to the σ model which incorporates PCAC. They then showed that one can preserve both of PCAC and gauge invariance if one uses a modification of the Gupta's[15] implementation of the Pauli-Villars regularization, but this spoils renormalizability.

Adler on the other hand imposed the gauge invariance on the vector vertices and examined what happens with the axial vector vertex. By this way he showed that the anomaly in the triangle diagrams is unavoidable in any sensible Lorentz invariant, local and renormalizable field theory with vector gauge symmetry. It has been established that the anomalous behavior of the triangle diagrams is in fact *necessary* to explain the main decay mode of the neutral pion into two photons in the soft pion limit.

It is instructive to see how the Pauli-Villars regularization for quark fields works in the analysis of the neutral pion decay and anomaly. If one denotes the axial-vector current and the pseudo-scalar density associated with the regulator field with mass M by $\tilde{A}_\mu(x)$ and $\tilde{P}(x)$, respectively, the identity (2.4) is replaced by

$$\partial^\mu(A_\mu(x) + \tilde{A}_\mu(x)) = 2miP(x) + 2Mi\tilde{P}(x). \quad (3.1)$$

For the massive quarks with $m \neq 0$, which breaks chiral symmetry explicitly, the pion is also massive. In the soft-pion limit with $p_\mu \rightarrow 0$ where p_μ is the four-momentum carried by the pion, the left-hand side of the above equation (3.1) goes to zero, and one obtains

$$\begin{aligned} \lim_{p_\mu \sim 0} \int dx e^{ipx} 2miP(x) &= \lim_{p_\mu \sim 0} - \int dx e^{ipx} 2Mi\tilde{P}(x) \\ &= - \lim_{p_\mu \sim 0} \int dx e^{ipx} \frac{e^2}{16\pi^2} \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta}(x). \end{aligned} \quad (3.2)$$

The left-hand side of this relation stands for the interpolating field for the soft-pion, while the right-hand side stands for the anomaly in the limit $M \rightarrow \infty$. The two-photon decay amplitude of the pion is thus correctly described.

In the Nambu's picture, one starts with the massless quarks with $m = 0$ and thus the ideal Nambu pion is also massless. In this case, one obtains from (3.1)

$$\begin{aligned} \lim_{p_\mu \sim 0} \int dx e^{ipx} \partial^\mu(A_\mu(x) + \tilde{A}_\mu(x)) &= \lim_{p_\mu \sim 0} \int dx e^{ipx} 2Mi\tilde{P}(x) \\ &= \lim_{p_\mu \sim 0} \int dx e^{ipx} \frac{e^2}{16\pi^2} \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta}(x). \end{aligned} \quad (3.3)$$

The left-hand side of this relation gives a non-vanishing matrix element between the vacuum and the Nambu pion due to the Nambu-Goldstone theorem, while the right-hand side stands for the anomaly in the limit $M \rightarrow \infty$. We thus obtain the correct pion decay amplitude.

The Pauli-Villars regularization was also successfully used by Bardeen [16] in the evaluation of the so-called non-Abelian gauge anomaly which satisfies the Wess-Zumino integrability condition[17].

In the revival of field theory, in particular, local gauge theory starting at the beginning of 1970s, 't Hooft and Veltman[18] introduced the dimensional regularization. With this regularization, we can now handle the vacuum polarization tensor without any ambiguity. On the other hand, one has a difficulty to handle the axial-vector current in this dimensional regularization. This regularization is based on the dimensional continuation of the algebra

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu} \quad (3.4)$$

but no consistent dimensional continuation of γ_5 which satisfies

$$\gamma_5 \gamma^\mu + \gamma^\mu \gamma_5 = 0. \quad (3.5)$$

is known. One thus understands the inevitable appearance of the triangle (or more generally *chiral*) anomaly, while one can preserve the vector gauge invariance consistently.

Starting around the early 1970s, there appeared many interesting applications of anomalies, which are nicely summarized in refs.[2][3]. During this period, the topological properties of the chiral anomaly were recognized and their applications have been discussed. In the presence of the instantons[19], the Atiyah-Singer index theorem[20] states the relation

$$n_+ - n_- = \nu \quad (3.6)$$

where n_\pm stand for the zero eigenvalue solutions of the Euclidean Dirac equation

$$\not{D}\phi_n(x) = 0 \quad (3.7)$$

with the simultaneous chiral eigenvalues

$$\gamma_5 \phi_n(x) = \pm \phi_n(x) \quad (3.8)$$

respectively. The ν in the right-hand of the above equation (3.6) stands for the Pontryagin number (or instanton number)

$$\nu = \frac{1}{16\pi^2} \int d^4x \text{tr} F \tilde{F} \quad (3.9)$$

The Atiyah-Singer index theorem is rigorously proved in the compact manifold such as S^4 . In the context of Euclidean flat space-time, the above index relation has been analyzed in detail by Jackiw and Rebbi[21].

As an interesting and far-reaching application of the topological properties of the chiral anomaly, 't Hooft[22] pointed out that the proton can decay even in the standard Weinberg-Salam theory. In fact the fermion number contains an anomaly for general parity violating Yang-Mills coupling in the presence of instantons.

4 Path integral formulation of anomalies

As a convenient means to relate the basically classical analysis of the Atiyah-Singer index theorem (3.6) and the quantum field theory, a path integral formulation of quantum anomaly was proposed[23]. In particular, the chiral anomaly was identified with the non-trivial Jacobian factor under the chiral transformation of fermionic variables.

To illustrate the path integral formulation, we start with the QCD-type Euclidean path integral

$$\int \mathcal{D}\bar{\psi}\mathcal{D}\psi[\mathcal{D}A_\mu] \exp\left[\int \bar{\psi}(i \not{D} - m)\psi d^4x + S_{YM}\right] \quad (4.1)$$

where γ^μ matrices are anti-hermitian with $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$, and $\gamma_5 = -\gamma^1\gamma^2\gamma^3\gamma^4$ is hermitian. The covariant derivative is defined by

$$\not{D} \equiv \gamma^\mu(\partial_\mu - igA_\mu^a T^a) = \gamma^\mu(\partial_\mu - igA_\mu) \quad (4.2)$$

with Yang-Mills generators T^a . S_{YM} stands for the Yang-Mills action and $[\mathcal{D}A_\mu]$ contains a suitable gauge fixing.

To analyze the chiral Jacobian we expand the fermion variables

$$\begin{aligned} \psi(x) &= \sum_n a_n \varphi_n(x) \\ \bar{\psi}(x) &= \sum_n \bar{b}_n \varphi_n^\dagger(x) \end{aligned} \quad (4.3)$$

in terms of the eigen-functions of hermitian \not{D}

$$\begin{aligned} \not{D}\varphi_n(x) &= \lambda_n \varphi_n(x) \\ \int d^4x \varphi_n^\dagger(x) \varphi_l(x) &= \delta_{n,l} \end{aligned} \quad (4.4)$$

which diagonalize the fermionic action in (4.1). The fermionic path integral measure is then written as

$$\mathcal{D}\bar{\psi}\mathcal{D}\psi = \lim_{N \rightarrow \infty} \prod_{n=1}^N d\bar{b}_n da_n \quad (4.5)$$

Under an infinitesimal global chiral transformation

$$\delta\psi = i\alpha\gamma_5\psi, \quad \delta\bar{\psi} = \bar{\psi}i\alpha\gamma_5 \quad (4.6)$$

we obtain the Jacobian factor

$$\begin{aligned} J &= \exp\left[-2i\alpha \lim_{N \rightarrow \infty} \sum_{n=1}^N \int d^4x \varphi_n^\dagger(x) \gamma_5 \varphi_n(x)\right] \\ &= \exp[-2i\alpha(n_+ - n_-)] \end{aligned} \quad (4.7)$$

where n_\pm stand for the number of eigenfunctions with vanishing eigenvalues and $\gamma_5\varphi_n = \pm\varphi_n$ in (4.4). We here used the relation

$$\int d^4x \varphi_n^\dagger(x) \gamma_5 \varphi_n(x) = 0 \quad (4.8)$$

for $\lambda_n \neq 0$ because $\not{D}\gamma_5\varphi_n(x) = -\lambda_n\gamma_5\varphi_n(x)$. The Atiyah-Singer index theorem $n_+ - n_- = \nu$ with Pontryagin index ν in (3.9), which was confirmed for one- instanton sector in R^4 space by Jackiw and Rebbi[21], shows that the chiral Jacobian (4.7) contains the correct information of chiral anomaly.

To extract a local version of the index (i.e., anomaly), we start with the expression

$$\begin{aligned} n_+ - n_- &= \lim_{N \rightarrow \infty} \sum_{n=1}^N \int d^4x \varphi_n^\dagger(x) \gamma_5 f((\lambda_n)^2/M^2) \varphi_n(x) \\ &= \lim_{N \rightarrow \infty} \sum_{n=1}^N \int d^4x \varphi_n^\dagger(x) \gamma_5 f(\not{D}^2/M^2) \varphi_n(x) \\ &\equiv \text{Tr} \gamma_5 f(\not{D}^2/M^2) \end{aligned} \quad (4.9)$$

for *any* smooth function $f(x)$ which rapidly goes to zero for $x = \infty$ with $f(0) = 1$. Since $\gamma_5 f(\not{D}^2/M^2)$ is a well-regularized operator, we may now use the plane wave basis of fermionic variables to extract an explicit gauge field dependence, and we define a *local version* of the index as

$$\begin{aligned} &\lim_{M \rightarrow \infty} \text{tr} \gamma_5 f(\not{D}^2/M^2) \\ &\equiv \lim_{M \rightarrow \infty} \sum_{n=1}^{\infty} \varphi_n^\dagger(x) \gamma_5 f(\not{D}^2/M^2) \varphi_n(x) \\ &= \lim_{M \rightarrow \infty} \text{tr} \int \frac{d^4k}{(2\pi)^4} e^{-ikx} \gamma_5 f(\not{D}^2/M^2) e^{ikx} \\ &= \lim_{M \rightarrow \infty} \text{tr} \int \frac{d^4k}{(2\pi)^4} \gamma_5 f\{(ik_\mu + D_\mu)^2/M^2 - \frac{ig}{4}[\gamma^\mu, \gamma^\nu]F_{\mu\nu}/M^2\} \\ &= \lim_{M \rightarrow \infty} \text{tr} M^4 \int \frac{d^4k}{(2\pi)^4} \gamma_5 f\{(ik_\mu + D_\mu/M)^2 - \frac{ig}{4}[\gamma^\mu, \gamma^\nu]F_{\mu\nu}/M^2\} \\ &= \frac{g^2}{32\pi^2} \text{tr} \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta} \end{aligned} \quad (4.10)$$

after a power expansion in $1/M$ [23]. We here used the relation

$$\not{D}^2 = D_\mu D^\mu - \frac{ig}{4}[\gamma^\mu, \gamma^\nu]F_{\mu\nu} \quad (4.11)$$

and the rescaling of the variable $k_\mu \rightarrow Mk_\mu$.

When one combines (4.9) and (4.10), one establishes the Atiyah-Singer index theorem (in R^4 space)

$$n_+ - n_- = \int d^4x \frac{g^2}{32\pi^2} \text{tr} \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta}. \quad (4.12)$$

We note that the local version of the index (anomaly) in (4.10) is valid for Abelian theory also.

From a view point of regularization, we note that the global index (4.9) as well as a local version of the index (4.10) are both independent of the regulator $f(x)$ provided [23]

$$f(0) = 1, \quad f(\infty) = 0, \quad f'(x)x|_{x=0} = f'(x)x|_{x=\infty} = 0. \quad (4.13)$$

Our regulator $f(x)$ imposes gauge invariance, and thus the regulator independence of chiral anomaly is consistent with the analysis of Adler[13], who showed that the chiral anomaly is independent of divergence and perfectly finite and well-defined if one imposes gauge invariance on the triangle diagram. From the definition of the basic path integral measure in (4.5), the present regularization may be called a *gauge invariant mode cut-off* regularization.

The Pauli-Villars regularization is realized in path integral formulation by rewriting the fermionic part of the path integral in (4.1) as

$$\int \mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}\bar{\phi}\mathcal{D}\phi \exp\left[\int \bar{\psi}(i\not{D} - m)\psi d^4x + \int \bar{\phi}(i\not{D} - M)\phi d^4x\right] \quad (4.14)$$

with a *bosonic* spinor ϕ . In this case, the Jacobian factors cancel among the contributions from ψ and ϕ , and thus the path integral measure becomes invariant under the chiral transformation. The (hard) chiral symmetry breaking by the mass term M of the regulator field ϕ in the limit $M \rightarrow \infty$ gives rise to the correct chiral anomaly. This fact was used in Section 3.

We note that the above path integral formulation works for the conformal (or Weyl) anomaly also[24]. A particularly elegant application of the Weyl anomaly was given by Polyakov in the context of the first quantization of string theory[25].

Among the applications of chiral anomaly, the existence of the anomaly even in the Einstein's general coordinate transformation which was shown by Alvarez-Gaume and Witten[26] enormously influenced our thinking about the quantum gravity and string theory. Other interesting applications of chiral and Weyl anomalies are found in [2][3].

5 Chiral anomalies in lattice gauge theory

The lattice theory provides a very powerful regularization of path integral. We have recently seen an impressive progress in the treatment of fermions and chiral anomaly in lattice gauge theory. This progress is based on the so-called Ginsparg-Wilson relation[27] and an explicit construction of lattice Dirac operator by Neuberger[28], which is called as the overlap Dirac operator for historical reasons. The overlap Dirac operator satisfies the Ginsparg-Wilson relation identically and it is free of species doubling. Hasenfratz, Laliena and Niedermayer proposed an interesting notion of the index theorem in lattice gauge theory[29] for the Dirac operator satisfying the Ginsparg-Wilson relation, which was in turn used by Lüscher[30] to identify a modified but exact chiral symmetry of lattice fermions. By these developments, we can now formulate the chiral anomaly for lattice theory in exactly the same manner as the continuum path integral. In particular, the chiral anomaly is defined as a non-trivial Jacobian in lattice theory also. Some of the manipulations become better defined in finite lattice theory, though certain aspects of topological considerations become more subtle on the discrete lattice space.

We would like to briefly summarize the essence of the lattice formulation of chiral anomaly. The lattice fermionic path integral is defined by

$$\int \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp\left[\int \bar{\psi}D\psi\right] \quad (5.1)$$

where the action is defined as a sum over the Euclidean hypercubic lattice points.

5.1 Representation of the Ginsparg-Wilson algebra

We start with the lattice Dirac operator D which satisfies the algebraic relation[31]

$$\gamma_5(\gamma_5 D) + (\gamma_5 D)\gamma_5 = 2a^{2k+1}(\gamma_5 D)^{2k+2} \quad (5.2)$$

where k stands for a non-negative integer and $k = 0$ corresponds to the customary Ginsparg-Wilson relation[30]. We here work on this general form of closed algebra (5.2), since it is known that we can construct a generalization of the overlap lattice Dirac operator, which is free of species doubling, for any value of k [31]. The parameter a stands for the lattice spacing. When one defines

$$H \equiv \gamma_5 a D \quad (5.3)$$

(5.2) is rewritten as

$$\gamma_5 H + H \gamma_5 = 2H^{2k+2} \quad (5.4)$$

or equivalently

$$\Gamma_5 H + \Gamma_5 H = 0 \quad (5.5)$$

where we defined

$$\Gamma_5 \equiv \gamma_5 - H^{2k+1}. \quad (5.6)$$

Note that both of H and Γ_5 are hermitian operators; in Euclidean lattice theory, D itself cannot be hermitian.

We now discuss a general representation of the algebraic relation (5.4). The relation (5.4) suggests that if

$$H\phi_n = a\lambda_n\phi_n, \quad (\phi_n, \phi_n) = 1 \quad (5.7)$$

with a real eigenvalue $a\lambda_n$ for the hermitian operator H , then

$$H(\Gamma_5\phi_n) = -a\lambda_n(\Gamma_5\phi_n). \quad (5.8)$$

Namely, the eigenvalues λ_n and $-\lambda_n$ are always paired if $\lambda_n \neq 0$ and $(\Gamma_5\phi_n, \Gamma_5\phi_n) \neq 0$. We also note the relation, which is derived by sandwiching the relation (5.4) by ϕ_n ,

$$(\phi_n, \gamma_5\phi_n) = (a\lambda_n)^{2k+1} \quad for \quad \lambda_n \neq 0. \quad (5.9)$$

Consequently

$$|(a\lambda_n)^{2k+1}| = |(\phi_n, \gamma_5\phi_n)| \leq ||\phi_n|| ||\gamma_5\phi_n|| = 1. \quad (5.10)$$

Namely, all the possible eigenvalues are bounded by

$$|\lambda_n| \leq \frac{1}{a}. \quad (5.11)$$

We thus evaluate the norm of $\Gamma_5\phi_n$

$$\begin{aligned}
(\Gamma_5\phi_n, \Gamma_5\phi_n) &= (\phi_n, (\gamma_5 - H^{2k+1})(\gamma_5 - H^{2k+1})\phi_n) \\
&= (\phi_n, (1 - H^{2k+1}\gamma_5 - \gamma_5 H^{2k+1} + H^{2(2k+1)})\phi_n) \\
&= [1 - (a\lambda_n)^{2(2k+1)}] \\
&= [1 - (a\lambda_n)^2][1 + (a\lambda_n)^2 + \dots + (a\lambda_n)^{4k}]
\end{aligned} \tag{5.12}$$

where we used (5.9). By remembering that all the eigenvalues are real, we find that ϕ_n is a “highest” state

$$\Gamma_5\phi_n = 0 \tag{5.13}$$

only if

$$[1 - (a\lambda_n)^2] = (1 - a\lambda_n)(1 + a\lambda_n) = 0 \tag{5.14}$$

for the Euclidean positive definite inner product $(\phi_n, \phi_n) \equiv \sum_x \phi_n^\dagger(x)\phi_n(x)$.

We thus conclude that the states ϕ_n with $\lambda_n = \pm\frac{1}{a}$ are *not* paired by the operation $\Gamma_5\phi_n$ and

$$\gamma_5 D\phi_n = \pm\frac{1}{a}\phi_n, \quad \gamma_5\phi_n = \pm\phi_n \tag{5.15}$$

respectively. These eigenvalues are in fact the maximum or minimum of the possible eigenvalues of H/a due to (5.11).

As for the vanishing eigenvalues $H\phi_n = 0$, we find from (5.4) that $H\gamma_5\phi_n = 0$, namely, $H[(1 \pm \gamma_5)/2]\phi_n = 0$. We thus have

$$\gamma_5 D\phi_n = 0, \quad \gamma_5\phi_n = \phi_n \quad \text{or} \quad \gamma_5\phi_n = -\phi_n. \tag{5.16}$$

To summarize the analyses so far, all the normalizable eigenstates ϕ_n of $\gamma_5 D = H/a$ are categorized into the following 3 classes:

(i) n_\pm (“zero modes”),

$$\gamma_5 D\phi_n = 0, \quad \gamma_5\phi_n = \pm\phi_n, \tag{5.17}$$

(ii) N_\pm (“highest states”),

$$\gamma_5 D\phi_n = \pm\frac{1}{a}\phi_n, \quad \gamma_5\phi_n = \pm\phi_n, \quad \text{respectively}, \tag{5.18}$$

(iii) “paired states” with $0 < |\lambda_n| < 1/a$,

$$\gamma_5 D\phi_n = \lambda_n\phi_n, \quad \gamma_5 D(\Gamma_5\phi_n) = -\lambda_n(\Gamma_5\phi_n). \tag{5.19}$$

Note that $\Gamma_5(\Gamma_5\phi_n) \propto \phi_n$ for $0 < |\lambda_n| < 1/a$.

We thus obtain the index relation[29][30]

$$\begin{aligned}
Tr\Gamma_5 &\equiv \sum_n (\phi_n, \Gamma_5 \phi_n) \\
&= \sum_{\lambda_n=0} (\phi_n, \Gamma_5 \phi_n) + \sum_{0 < |\lambda_n| < 1/a} (\phi_n, \Gamma_5 \phi_n) + \sum_{|\lambda_n|=1/a} (\phi_n, \Gamma_5 \phi_n) \\
&= \sum_{\lambda_n=0} (\phi_n, \Gamma_5 \phi_n) \\
&= \sum_{\lambda_n=0} (\phi_n, (\gamma_5 - H^{2k+1})\phi_n) \\
&= \sum_{\lambda_n=0} (\phi_n, \gamma_5 \phi_n) \\
&= n_+ - n_- = index
\end{aligned} \tag{5.20}$$

where n_{\pm} stand for the number of normalizable zero modes with $\gamma_5 \phi_n = \pm \phi_n$ in the classification (i) above. We here used the fact that $\Gamma_5 \phi_n = 0$ for the “highest states” and that ϕ_n and $\Gamma_5 \phi_n$ are orthogonal to each other for $0 < |\lambda_n| < 1/a$ since they have eigenvalues with opposite signatures.

We note that all the states ϕ_n with $0 < |\lambda_n| < 1/a$, which appear pairwise with $\lambda_n = \pm |\lambda_n|$, can be normalized to satisfy the relations

$$\begin{aligned}
\Gamma_5 \phi_n &= [1 - (a\lambda_n)^{2(2k+1)}]^{1/2} \phi_{-n}, \\
\gamma_5 \phi_n &= (a\lambda_n)^{2k+1} \phi_n + [1 - (a\lambda_n)^{2(2k+1)}]^{1/2} \phi_{-n}.
\end{aligned} \tag{5.21}$$

Here ϕ_{-n} stands for the eigenstate with an eigenvalue opposite to that of ϕ_n . These states ϕ_n cannot be the eigenstates of γ_5 since $|(\phi_n, \gamma_5 \phi_n)| = |(a\lambda_n)^{2k+1}| < 1$.

5.2 Chiral Jacobian in lattice theory

The Euclidean path integral for a fermion is defined by

$$\int \mathcal{D}\bar{\psi} \mathcal{D}\psi \exp\left[\int \bar{\psi} D\psi\right] \tag{5.22}$$

where

$$\int \bar{\psi} D\psi \equiv \sum_{x,y} \bar{\psi}(x) D(x,y) \psi(y) \tag{5.23}$$

and the summation runs over all the points on the lattice. The relation (5.4) is re-written as

$$\gamma_5 \Gamma_5 \gamma_5 D + D \Gamma_5 = 0 \tag{5.24}$$

and thus the Euclidean action is invariant under the global “chiral” transformation[30]

$$\begin{aligned}
\bar{\psi}(x) &\rightarrow \bar{\psi}'(x) = \bar{\psi}(x) + i \sum_z \bar{\psi}(z) \epsilon \gamma_5 \Gamma_5(z,x) \gamma_5 \\
\psi(y) &\rightarrow \psi'(y) = \psi(y) + i \sum_w \epsilon \Gamma_5(y,w) \psi(w)
\end{aligned} \tag{5.25}$$

with an infinitesimal constant parameter ϵ . Under this transformation, one obtains a Jacobian factor

$$\mathcal{D}\bar{\psi}'\mathcal{D}\psi' = J\mathcal{D}\bar{\psi}\mathcal{D}\psi \quad (5.26)$$

with

$$J = \exp[-2i\text{Tr}\epsilon\Gamma_5] = \exp[-2i\epsilon(n_+ - n_-)] \quad (5.27)$$

where we used the index relation (5.20).

We now relate this index appearing in the Jacobian to the Pontryagin index of the gauge field in a smooth continuum limit. We start with

$$\text{Tr}\{\Gamma_5 f(\frac{(\gamma_5 D)^2}{M^2})\} = \text{Tr}\{\Gamma_5 f(\frac{(H/a)^2}{M^2})\} = n_+ - n_- \quad (5.28)$$

Namely, the index is not modified by any regulator $f(x)$ with $f(0) = 1$ and $f(x)$ rapidly going to zero for $x \rightarrow \infty$, as can be confirmed by using (5.20). This means that you can use *any* suitable $f(x)$ in the evaluation of the index by taking advantage of this property. We then consider a local version of the index

$$\text{tr}\{\Gamma_5 f(\frac{(\gamma_5 D)^2}{M^2})\}(x, x) = \text{tr}\{(\gamma_5 - H^{2k+1})f(\frac{(\gamma_5 D)^2}{M^2})\}(x, x) \quad (5.29)$$

where trace stands for Dirac and Yang-Mills indices; Tr in (5.28) includes a sum over the lattice points x . A local version of the index is not sensitive to the precise boundary condition, and one may take an infinite volume limit of the lattice in the above expression.

We now examine the continuum limit $a \rightarrow 0$ of the above local expression (5.29). (This continuum limit corresponds to the so-called “naive” continuum limit in the context of lattice gauge theory.) We first observe that the term

$$\text{tr}\{H^{2k+1}f(\frac{(\gamma_5 D)^2}{M^2})\} \quad (5.30)$$

goes to zero in this limit. The large eigenvalues of $H = a\gamma_5 D$ are truncated at the value $\sim aM$ by the regulator $f(x)$ which rapidly goes to zero for large x .

We thus examine the small a limit of

$$\text{tr}\{\gamma_5 f(\frac{(\gamma_5 D)^2}{M^2})\}. \quad (5.31)$$

The operator appearing in this expression is well regularized by the function $f(x)$, and we evaluate the above trace by using the plane wave basis to extract an explicit gauge field dependence. We consider a square lattice where the momentum is defined in the Brillouin zone

$$-\frac{\pi}{2a} \leq k_\mu < \frac{3\pi}{2a}. \quad (5.32)$$

We assume that the operator D is free of species doubling; in other words, the operator D blows up rapidly ($\sim \frac{1}{a}$) for small a in the momentum region corresponding to species

doublers. The contributions of doublers are eliminated by the regulator $f(x)$ in the above expression, since

$$\text{tr}\{\gamma_5 f(\frac{(\gamma_5 D)^2}{M^2})\} \sim (\frac{1}{a})^4 f(\frac{1}{(aM)^2}) \rightarrow 0 \quad (5.33)$$

for $a \rightarrow 0$ if one chooses $f(x) = e^{-x}$, for example.

We thus examine the above trace in the momentum range of the physical species

$$-\frac{\pi}{2a} \leq k_\mu < \frac{\pi}{2a}. \quad (5.34)$$

We obtain the limiting $a \rightarrow 0$ expression[31]

$$\begin{aligned} & \lim_{a \rightarrow 0} \text{tr}\{\gamma_5 f(\frac{(\gamma_5 D)^2}{M^2})\}(x, x) \\ &= \lim_{a \rightarrow 0} \text{tr} \int_{-\frac{\pi}{2a}}^{\frac{\pi}{2a}} \frac{d^4 k}{(2\pi)^4} e^{-ikx} \gamma_5 f(\frac{(\gamma_5 D)^2}{M^2}) e^{ikx} \\ &= \lim_{L \rightarrow \infty} \lim_{a \rightarrow 0} \text{tr} \int_{-L}^L \frac{d^4 k}{(2\pi)^4} e^{-ikx} \gamma_5 f(\frac{(\gamma_5 D)^2}{M^2}) e^{ikx} \\ &= \lim_{L \rightarrow \infty} \text{tr} \int_{-L}^L \frac{d^4 k}{(2\pi)^4} e^{-ikx} \gamma_5 f(\frac{(i\gamma_5 \not{D})^2}{M^2}) e^{ikx} \\ &\equiv \text{tr}\{\gamma_5 f(\frac{\not{D}^2}{M^2})\} \end{aligned} \quad (5.35)$$

where we first take the limit $a \rightarrow 0$ with fixed k_μ in $-L \leq k_\mu \leq L$, and then take the limit $L \rightarrow \infty$. This procedure is justified if the integral is well convergent. We also *assumed* that the operator D satisfies the following relation in the limit $a \rightarrow 0$

$$\begin{aligned} D e^{ikx} h(x) &\rightarrow e^{ikx} (-\not{k} + i \not{\partial} - g \not{A}) h(x) \\ &= i(\not{\partial} + ig \not{A})(e^{ikx} h(x)) \equiv i \not{D}(e^{ikx} h(x)) \end{aligned} \quad (5.36)$$

for any *fixed* k_μ , $(-\frac{\pi}{2a} < k_\mu < \frac{\pi}{2a})$, and a sufficiently smooth function $h(x)$. The function $h(x)$ corresponds to the gauge potential in our case, which in turn means that the gauge potential $A_\mu(x)$ is assumed to vary very little over the distances of the elementary lattice spacing.

The condition (5.36) as well as the absence of species doubling are satisfied by the overlap Dirac operator[28] and its generalization[31]. The last expression in (5.35) is identical to the continuum result (4.10). We thus obtain from (5.28) and (5.35) the lattice index relation

$$n_+ - n_- = \text{Tr}\{\Gamma_5 f(\frac{(\gamma_5 D)^2}{M^2})\} = \int d^4 x \frac{g^2}{32\pi^2} \text{tr} \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta} \quad (5.37)$$

in the continuum limit. A local version of this relation leads to the lattice evaluation of chiral anomaly.

5.3 Fermion number anomaly in chiral lattice gauge theory

As for the chiral fermions on the lattice, the general algebra (5.2) satisfies the decomposition

$$D = \frac{(1 + \gamma_5)}{2} D \frac{(1 - \hat{\gamma}_5)}{2} + \frac{(1 - \gamma_5)}{2} D \frac{(1 + \hat{\gamma}_5)}{2} \quad (5.38)$$

with

$$\hat{\gamma}_5 \equiv \gamma_5 - 2a^{2k+1}(\gamma_5 D)^{2k+1}, \quad (\hat{\gamma}_5)^2 = 1 \quad (5.39)$$

by noting $\gamma_5(\gamma_5 D)^2 = (\gamma_5 D)^2 \gamma_5$ which can be proved by using (5.2).

The fermion number non-conservation in the chiral theory defined by

$$\begin{aligned} & \int \mathcal{D}\bar{\psi} \mathcal{D}\psi \exp\left\{\int \bar{\psi} D_L \psi\right\} \\ & \equiv \int \mathcal{D}\bar{\psi} \mathcal{D}\psi \exp\left\{\int \bar{\psi} \frac{(1 + \gamma_5)}{2} D \frac{(1 - \hat{\gamma}_5)}{2} \psi\right\} \end{aligned} \quad (5.40)$$

follows from the fermion number transformation

$$\psi \rightarrow e^{i\alpha} \psi, \quad \bar{\psi} \rightarrow \bar{\psi} e^{-i\alpha}. \quad (5.41)$$

If one remembers that the functional spaces of the variables ψ and $\bar{\psi}$ are specified by the *projection operators* $(1 - \hat{\gamma}_5)/2$ and $(1 + \gamma_5)/2$, respectively, the Jacobian factor for the transformation (5.41) is given by[30]

$$\begin{aligned} J &= \exp\left\{i\alpha \text{Tr}\left[\frac{(1 + \gamma_5)}{2} - \frac{(1 - \hat{\gamma}_5)}{2}\right]\right\} \\ &= \exp\left\{i\alpha \text{Tr}[\gamma_5 - (\gamma_5 a D)^{2k+1}]\right\} = \exp\{i\alpha[n_+ - n_-]\} \end{aligned} \quad (5.42)$$

where the index is defined in (5.20). (To be precise, the lattice formulation of chiral non-Abelian gauge theory has not been established yet, but the chiral $U(1)$ anomaly in (5.42) is evaluated without knowing the details of the path integral measure of chiral gauge theory.)

We thus reproduce the well-known fermion number non-conservation in chiral non-Abelian theory[22].

6 Conclusion

We have briefly reviewed some aspects of anomalies with an emphasis on the regularization of field theory. We discussed the recent development in lattice theory in some detail, since this subject is relatively new. As for more extensive reviews of anomalies, the readers are asked to look at the review articles such as [1][2] [3].

References

- [1] S.L. Adler, in *Lectures on Elementary Particles and Quantum Field Theory*, edited by S. Deser et al. (MIT Press, Cambridge, Mass., 1970).

- [2] S.B. Treiman, R. Jackiw, B. Zumino and E. Witten, *Current Algebra and Anomalies* (World Scientific, Singapore, 1985).
- [3] R. Bertlmann, *Anomalies in Quantum Field Theory* (Oxford University Press, Oxford, 1996).
- [4] S. Tomonaga, Phys. Rev. **74** (1948) 224.
- [5] Y. Miyamoto, Private communication.
- [6] H. Fukuda and Y. Miyamoto, Prog. Theor. Phys.**4** (1949) 49.
- [7] W. Pauli and F. Villars, Rev. Mod. Phys. **21** (1949) 434.
- [8] H. Fukuda, Y. Miyamoto, T. Miyajima, S. Tomonaga, S. Oneda and S. Sasaki, Prog. Theor. Phys. **4** (1949) 477.
- [9] J. Steinberger, Phys. Rev. **76** (1949) 1180.
- [10] J. Schwinger, Phys. Rev.**82** (1951) 664.
- [11] K. Nishijima, Bull. Phys. Soc. Jpn.,**47** (1992) 859.
- [12] J.S. Bell and R. Jackiw, Nuovo Cim. **60A** (1969) 47.
- [13] S.L. Adler, Phys. Rev. **177** (1969) 2426.
- [14] Y. Nambu and G. Jona-Lasinio, Phys. Rev.**122** (1961) 345.
- [15] S.N. Gupta, Proc. Phys. Soc., **A66** (1953) 129.
- [16] W.A. Bardeen, Phys. Rev. **184** (1969) 1848.
- [17] J. Wess and B. Zumino, Phys. Lett. **37B** (1971) 95.
- [18] G. 't Hooft and M. Veltman, Nucl. Phys. **B44**(1972) 189.
C.G. Bollini and J.J. Giambiagi, Nuovo Cim. **B12** (1972) 20.
- [19] A.A. Belavin, A.M. Polyakov, A.S. Schwartz and Yu.S. Tyupkin, Phys. Lett. **59B** (1975) 85.
- [20] M. Atiyah, R. Bott, and V. Patodi, Invent. Math. **19**(1973)279.
- [21] R. Jackiw and C. Rebbi, Phys. Rev.**D16**(1977)1052.
- [22] G. 't Hooft, Phys. Rev. Lett. **37** (1976) 8.
- [23] K. Fujikawa, Phys. Rev. Lett. **42** (1979) 1195; Phys. Rev. **D21** (1980) 2848; **D22** (1980) 1499(E).
- [24] K. Fujikawa, Phys. Rev. Lett. **44**(1980)1733 .

- [25] A.M. Polyakov, Phys. Lett. **103B**(1981) 207.
- [26] L. Alvarez-Gaume and E. Witten, Nucl. Phys. **B234** (1983) 269.
- [27] P.H. Ginsparg and K.G. Wilson, Phys. Rev. **D25** (1982)2649.
- [28] H. Neuberger, Phys. Lett.**B417**(1998)141;**B427**(1998)353.
- [29] P. Hasenfratz, V. Laliena and F. Niedermayer, Phys. Lett. **B427**(1998)125, and references therein.
- [30] M. Lüscher, Phys.Lett.**B428** (1998) 342; Nucl.Phys. **B549** (1999) 295.
- [31] K. Fujikawa, “Algebraic generalization of the Ginsparg-Wilson relation”, hep-lat/0004012 (to appear in Nucl. Phys. B).